N74-32038

WELL-POSED CAUCHY PROBLEMS

Heinz-Otto Kreiss

1. Report No. NASA TT F-15,893	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle WELL-POSED CAUCHY PROBLEMS		5. Report Date August 1974
		6. Performing Organization Code
7. Author(s) Heinz-Otto Kreiss		8. Performing Organization Report No.
		10. Work Unit No.
9. Performing Organization Name and Address SCITRAN Box 5456 Santa Barbara, CA 93108		11. Contract or Grant No. NASw-2483
		13. Type of Report and Period Covered Translation
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

Discussion of partial differential equation systems whose coefficients depend on x and t. Problems equivalent to the original Cauchy problems are formulated and solved.

PRECEDING PAGE BLANK NOT FILMED

17. Key Words (Selected by Author(s))	18. Distribution Statement Unclassified - Unlimited

19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 30	22. Price

WELL-POSED CAUCHY PROBLEMS

Heinz-Otto Kreiss

1. INTRODUCTION

In this paper we shall consider systems of partial differential equations of the type

$$\partial u/\partial t = P(t, \partial/\partial x)u = \sum_{j=0}^{m} P_j(t, \partial/\partial x)u \qquad (1.1)$$

and shall investigate the question as to when the Cauchy problem is well-posed in an interval $0 \leq t \leq T, \ T > 0$. Here $x = (x_1, \ldots, x_s)'$ denote points in the real s-dimensional space R and $u = (u_1(x,t), \ldots u_n(x,t))'$ are function vectors in the complex n-dimensional space S and

$$P_j(t, \partial/\partial x)u = \sum_{|\nu|=j} A_\nu(t) \, \partial^{|\nu|} u/\partial x_1^{\nu_1} \ldots \partial x_s^{\nu_s},$$
$$\nu = (\nu_1, \ldots \nu_s), \qquad |\nu| = \sum \nu_i, \qquad (1.2)$$

are homogeneous differential operators of order j whose coefficients are continuous square matrices which depend on t. (y' denotes the transpose of the vector y.)

Furthermore, in the last section of this paper we shall apply the results obtained to the investigation of first order systems whose coefficients depend on x also. By so doing we are able to obtain "a priori" inequalities which provide a means of generalizing the results of Leray [5] and Petrovskii [7].

1

This paper depends essentially upon a previous paper of the author [4], and we assume familiarity with it.

2. DEFINITION OF APPROPRIATENESS

The well-posedness of the Cauchy problem for the system (1.1) has already been studied intensively by Petrovskii [8]. In that paper he defines well-posedness in the sense of Hadamard [3] essentially as follows: there exists a constant C and a natural number p such that for all t_1, t_2 where $0 \leq t_1 \leq t_2 \leq T$, the estimate

$$\sup_{x \in R} |u(x,t_2)| \leq C \sum_{j=0}^{p} \sum_{|\nu|=j} \sup_{x \in R} |\partial^\nu u(x,t_1)/\partial x_1^{\nu_1} \ldots \partial x_s^{\nu_s}| \qquad (2.1)$$

/110

where $|u|^2 = \Sigma |u_i|^2$ holds for the solutions of (1.1) Here all functions are admissible as boundary values for the system (1.1) at an arbitrary time t_1 which are differentiable sufficiently often, are bounded together with their derivatives, and for which a solution of the Cauchy problem exists. If we let

$$P(t,i\omega) = \sum_{j=0}^{m} P_j(t,i\omega), \quad P_j(t,i\omega) = \sum_{|\nu|=j} A_\nu(t)(i\omega_1)^{\nu_1} \ldots (i\omega_s)^{\nu_s}, \qquad (2.2)$$

then Petrovskii proves the following criterion:

The Cauchy problem (1.1) with constant coefficients is well-posed in the sense of Hadamard if and only if there are constants C_i such that the inequality

$$\mathrm{Re}\, \varkappa_j(\omega) \leq C_1 \log|1 + |\omega|| + C_2 \qquad (2.3)$$

holds for the eigenvalues $\varkappa_j(\omega)$ of $P(i\omega)$. (L. Gårding [2] proved subsequently that one can set $C_1 = 0$).

2

In general this criterion no longer holds when the coefficients of the differential operator depend on t. In fact one can present systems (1.1) which are not well-posed / in the sense of Hadamard, even though (2.3) holds for every $t \in (0,T)$. Such a system is

$$\frac{\partial u}{\partial t} = U(t) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} U^{-1}(t) \frac{\partial u}{\partial x}, \quad U(t) = \begin{pmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{pmatrix}. \tag{2.4}$$

Obviously (2.3) is satisfied for every t. If we now introduce $v = U^{-1}u$ as a new variable we obtain

$$\frac{\partial v}{\partial t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \frac{\partial v}{\partial x} - U^{-1} \frac{\partial U}{\partial t} v = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \frac{\partial v}{\partial x} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} v. \tag{2.5}$$

Since U is an orthogonal matrix, the Cauchy problem (2.4) is well-posed precisely when (2.5) is well-posed. But (2.5) is a system with constant coefficients. Therefore we can apply the criterion (2.3). Now

$$P(i\omega) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} i\omega - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and for the eigenvalues we have

$$\varkappa(\omega) = i\omega \pm [-(1+i\omega)]^{\frac{1}{2}}.$$

The inequality (2.3) is therefore not satisfied and thus the Cauchy problem (2.4) is not well-posed. (Note that corresponding considerations hold if one replaces the variable t in U (t) by x). Hence in order to obtain an algebraic criterion which also holds — under suitable continuity assumptions — for coefficients which depend on t, it is necessary to alter the definition of well-posedness.

/111

3

We now define what it is that we wish to understand by *a well-posed* Cauchy problem in this paper. Let L_2 be the Hilbert space of all n-dimensional vector functions $u(x)$, $v(x)$, ... in R whose absolute-square-is-integrable and whose norm and scalar product are defined as usual by

$$(u,v) = \int_R u^* v \, dx = \int_R \sum_{i=1}^{n} \bar{u}_i v_i \, dx, \quad (u,u) = \|u\|^2 ,$$ (2.6)

Furthermore we denote by $\mathfrak{M} \subset L_2$ the class of all vector functions $f(x)$ whose Fourier transforms

$$\varphi(\omega) = (2\pi)^{-s/2} \int_R e^{-i\omega x} f(x) \, dx, \quad \omega = (\omega_1, \ldots, \omega_s) \text{ real}$$ (2.7)

exist, which are piecewise continuous, and which vanish outside of a compact set $\subset R'$ (independent of f). Here R' denotes the real s-dimensional space of the variables $\omega = (\omega_1, \ldots \omega_s)$. For the Cauchy problem (1.1) we admit as initial values at arbitrary time $t_1 \varepsilon (0,T)$ all functions $f(x)$ which lie in $\mathfrak{M}$. If one applies the Fourier transform to the system (1.1), then by virtue of (2.7) one obtains

$$\psi(\omega,t) = (2\pi)^{-s/2} \int_R e^{-i\omega x} u(x,t) \, dx ,$$
$$d\psi(\omega,t)/dt = P(t,i\omega)\psi(\omega,t), \quad \psi(\omega,t_1) = \varphi(\omega) .$$ (2.8)

If one solves these systems of ordinary differential equations, then for each initial value distribution $f(x) \in \mathfrak{M}$ one obtains a solution

$$u(x,t) = (2\pi)^{-s/2} \int_{R'} e^{i\omega x} \psi(\omega,t) \, d\omega, \quad t \geq t_1 ,$$ (2.9)

of the Cauchy problem (1.1) which belongs to the set $\mathfrak{M}$ for each fixed t and which is continuous with respect to t and infinitely differentiable with respect to x. We now define:

DEFINITION 1. <u>The Cauchy problem is well-posed</u> /if there is a constant C such that for all t_1, t_2 where $0 \leq t_1 \leq t_2 \leq T$, and for all solutions (2.9) the inequality

$$\|u(x,t_2)\| \leq C \|u(x,t_1)\|$$ (2.10)

holds.

The inequality (2.10) is the decisive requirement for well-posedness. Indeed since $\mathfrak{M}$ is dense in L_2 one can admit all functions of L_2 as initial values if one introduces generalized solutions in the usual manner. For these solutions (2.10) then holds also.

3. CONDITIONS FOR THE SOLUTIONS OF THE DIFFERENTIAL EQUATIONS (2.8).

We shall now state necessary and sufficient conditions which the solutions of the differential equations (2.8) must satisfy in order for the Cauchy problem to be well-posed. We prove

THEOREM 1. <u>The Cauchy problem (1.1) is well-posed</u> if and only if there is a constant D>0 such that for all ω, for all t_1, t_2 with $0 \leq t_1 \leq t_2 \leq T$, and for all solutions of (2.8) the inequality

$$|\psi(\omega,t_2)| \leq D |\psi(\omega,t_1)|$$ (3.1)

holds.

PROOF. If (3.1) holds then it obviously follows from Parseval's equation that the Cauchy problem is well-posed. That the condition (3.1) is also necessary is proved as follows. Assume that (3.1) does not hold. Then there exist sequences

$\omega_\nu, t_{1\nu}, t_{2\nu}|$ and $\psi(\omega_\nu, \bar{t}_{1\nu})|$ with $0 \leqq t_{1\nu} \leqq t_{2\nu} \leqq T$ such that

$$|\psi(\omega_\nu, t_{2\nu})| \geqq \nu |\psi(\omega_\nu, t_{1\nu})|, \quad \nu = 1, 2, \dots. \tag{3.2}$$

Since the solutions of the differential equations (2.8) depend continuously on ω and on the initial values, there exists a $\delta_\nu > 0$ such that for all ω with $|\omega - \omega_\nu| \leqq \delta_\nu$ the following inequality holds.

$$|\psi(\omega, t_{2\nu})| \geqq \tfrac{1}{2}\nu |\psi(\omega, t_{1\nu})|| \text{ provided that } \psi(\omega, t_{1\nu}) = \psi(\omega_\nu, t_{1\nu}). \tag{3.3}$$

If we now let

$$\psi(\omega, t_{1\nu}) = \begin{cases} \psi(\omega_\nu, t_{1\nu}) & \text{for } |\omega - \omega_\nu| \leqq \delta_\nu \\ 0 & \text{for } |\omega - \omega_\nu| > \delta_\nu \end{cases}$$

be initial values for a solution $u_\nu(x, t)$ of the form (2.9) for the Cauchy problem at time $t = t_{1\nu}$, then

$$\|u_\nu(x, t_{2\nu})\| \geqq \tfrac{1}{2}\nu \|u_\nu(x, t_{1\nu})\|.$$

follows from (3.3) and Parseval's equation. Thus (2.10) is not satisfied, and the Cauchy problem is therefore not well-posed.

4. SYSTEMS (1.1) WITH CONSTANT COEFFICIENTS

We now consider systems (1.1) with constant coefficients and we shall state a necessary and sufficient algebraic criterion for the well-posedness of the Cauchy problem. First note that one can write the solutions of the systems of ordinary differential equations (2.8) in the form

$$\psi(\omega, t) = e^{P(i\omega)(t - t_1)} \psi(\omega, t_1) \tag{4.1}$$

Hence from Theorem 1 follows immediately

6

THEOREM 2. The Cauchy problem (1.1) with constant coefficients is well-posed if and only if there exists a constant D>0 such that for all ω

$$|e^{P(i\omega)t}| \leq D, \qquad 0 \leq t \leq T.$$ (4.2)

($|A|$ is the Euclidean norm of the matrix A, i.e., $|A| = \sup |Ax| / |x|$.)

If the inequality (4.2) holds, we can write it in a somewhat different form. If we let $(\log D)/T = \alpha$, it then follows that

$$|e^{P(i\omega)T}| \leq e^{\alpha T}.$$

Hence for $0 \leq t < \infty$ we have

$$|e^{P(i\omega)t}| \leq D e^{\alpha t},$$

and for the family $\mathscr{F}$ of matrices $P(i\omega) - \alpha I$ we have

$$|e^{(P(i\omega) - \alpha I)t}| \leq D, \qquad 0 \leq t < \infty.$$ (4.3)

Therefore, we can apply the main theorem of [6] to the family $\mathscr{F}$. This theorem provides necessary and sufficient criteria which the family $\mathscr{F}$ must satisfy in order that (4.3) shall hold. We obtain

THEOREM 3. The following statements are equivalent:

1) The Cauchy problem (1.1) with constant coefficients is well-posed.

2) There exist real constants C_{31}, C_{32} and α, and to each matrix $P(i\omega)$ there corresponds a matrix $S = S(\omega)$ satisfying

$$\max(|S|, |S^{-1}|) \leq C_{31}$$ (4.4)

so that

7

$$B = S(P(i\omega) - \alpha I)S^{-1}$$
$$\leq SP(i\omega)S^{-1} - \alpha I = \begin{pmatrix} \varkappa_1 - \alpha & b_{12} & \cdots\cdots & b_{1n} \\ 0 & \varkappa_2 - \alpha & b_{23} \cdots & b_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & \cdots\cdots\cdots & 0 & \varkappa_n - \alpha \end{pmatrix}$$

and the inequalities

$$\operatorname{Re}\varkappa_j - \alpha \leq \operatorname{Re}\varkappa_k - \alpha \leq 0, \quad j \geq k, \tag{4.5}$$

and

$$|b_{kj}| \leq C_{32}(|\operatorname{Re}\varkappa_k - \alpha|) \tag{4.6}$$

hold.

3) There exist real constants C_4 and α, and corresponding to each matrix $P(i\omega)$ there exists a positive definite hermitian matrix $H(\omega)$ for which

$$\max(|H|, |H^{-1}|) \leq C_4, \quad \text{that is} \quad C_4^{-1}I \leq H \leq C_4 I \tag{4.7}$$

so that

$$H(\omega)(P(i\omega) - \alpha I) + (P^*(i\omega) - \alpha I)H(\omega) \leq 0. \tag{4.8}$$

Furthermore it should be noted that according to section 4 of [4] one can construct the matrices $H(\omega)$ with the help of the matrices $S(\omega)$. In fact one can assume H in the form

$$H = S^*DS \tag{4.9}$$

where D is a suitable positive definite diagonal matrix independent of ω.

We now wish to formulate the third statement of the last theorem in another way. For this purpose we introduce

---

* Denotes the adjoint matrix of A.

DEFINITION 2. The differential operator $P(\partial/\partial x)$ is semi-bounded with respect to a norm (u, Hu) if

1) H is a linear bounded positive definite symmetric operator in L_2 which is defined everywhere, and

2) There exists a constant α such that for all $u(x) \in \mathfrak{M}$ [cf. (2.7)]:

$$(P(\partial/\partial x)u, Hu) + (u, HP(\partial/\partial x)u) = 2\,\mathrm{Re}(P(\partial/\partial x)u, Hu)$$
$$\leqq 2\alpha(u, Hu).$$

Note that the norm (u, Hu) is equivalent to the L_2-norm. We now prove

THEOREM 3. The Cauchy problem (1.1) with constant coefficients is well-posed if and only if $P(\partial/\partial x)$ is semi-bounded with respect to a norm $(u, H\omega)$.

PROOF. Let the Cauchy problem be well-posed. Then it follows from the proof of the main theorem of [4] (cf., Section 3) that one can assume that the matrices $S(\omega)$ — and therefore by (4.9) the matrices $H(\omega)$ of the previous theorem — are piecewise continuous functions of ω. Indeed one can write $S(\omega)$ in the form $S(\omega) - S_1(\omega) U(\omega)$ where $U(\omega)$ is any unitary matrix which transforms $P(i\omega)$ into triangular form (Schur's normal form), and the coefficients of $S_1(\omega)$ are fractional rational functions of the coefficients of $U(\omega)P(i\omega)U^*(\omega)$. Hence by means of /115

$$Hu = \int_{R'} e^{i\omega x} H(\omega)\psi(\omega)\,d\omega, \qquad u = (2\pi)^{-s/2}\int_{R'} e^{i\omega x}\psi(\omega)\,d\omega, \qquad (4.10)$$

we can define a linear bounded and positive definite symmetric operator H in L_2 for which the inequality

$$C_4^{-1}\|u\|^2 \leqq (u, Hu) = \int_{R'} \psi^*(\omega)H(\omega)\psi(\omega)\,d\omega \leqq C_4\|u\|^2.$$

holds by virtue of (4.7) and Parseval's equation. From the inequality (4.8) it then follows that for all $u(x) \in \mathfrak{M}$,

$$(P(\partial/\partial x)u, Hu) + (u, HP(\partial/\partial x)u)$$

$$= \int_{R'} \psi^*(P^*(i\omega)H(\omega) + H(\omega)P(i\omega))\psi \, d\omega$$

$$\leq 2\alpha \int_{R'} \psi^* H(\omega)\psi \, d\omega$$

$$= 2\alpha(u, Hu).$$

(4.11)

Conversely if $P(\partial/\partial x)$ is semi-bounded with respect to a norm (u, Hu) then

$$\begin{aligned}
\partial(u, Hu)/\partial t &= (\partial u/\partial t, Hu) + (u, H\partial u/\partial t) \\
&= (P(\partial/\partial x)u, Hu) + (u, HP(\partial/\partial x)u) \\
&\leq 2\alpha(u, Hu).
\end{aligned}$$

(4.12)

holds for all solutions of the Cauchy problem of the form (2.9) which belong to the set $\mathfrak{M}$ for each fixed t. Since the norm (u, Hu) is equivalent to the L_2-norm, the inequality (2.10) follows from (4.12). Thus the Cauchy problem is well-posed.

5. SYSTEMS (1.1) WITH COEFFICIENTS WHICH DEPEND ON t.

We shall now show that the two criteria of the last theorem hold in the main for coefficients which depend on t. We prove

THEOREM 4. Assume that the second or third statement of theorem 3 holds for all fixed $t \in (0,T)$ where the constants C_{3i} and α or C_4 and α are independent of t. Then the Cauchy problem (1.1) is well-posed if the matrices $S(t,\omega)$ or $H(t,\omega)$ are continuously differentiable with respect to t for each fixed ω, and there is a constant K independent of t and ω so that $|\partial S/\partial t| \leq K$ or $\partial H/\partial t \leq KI$.

10

PROOF. If the second statement of Theorem 3 holds for constants C_{31}, α and if $S(t,\omega)$ is continuously differentiable, then it follows from section 4 of [3], applied to the family $\mathscr{F}$ of matrices $P(t,i\omega) - \alpha I, \omega$ real and $t \in (0,T)$, that there is a constant positive definite diagonal matrix D such that the matrices

$$H = S^*DS$$

have the properties assumed in the theorem. Hence we only need to prove the theorem for the case in which there are such matrices $H(t,\omega)$. If we now consider the ordinary differential equations (2.8) it follows from (4.7) and (4.8) that

$$d\psi^*H\psi/dt = \psi^*(HP + P^*H)\psi + \psi^*(\partial H/\partial t)\psi$$
$$\leq (2\alpha + C_4 K)\psi^*H\psi.$$

$$(5.1)$$

Thus the inequality (3.1) is satisfied, and therefore by Theorem 1, the Cauchy problem is well-posed.

6. EXAMPLES OF WELL-POSED CAUCHY PROBLEMS

We shall now mention some examples of well-posed Cauchy problems.

1) Parabolic Systems (cf. Petrovskii [8]).

If $m \equiv 0 \pmod 2$ and if there is a constant $\delta > 0$ such that for all ω, t the inequality

$$\mathrm{Re}\,x_j(\omega) \leq -2\delta|\omega|^m + \delta^{-1}$$

$$(6.1)$$

holds for the eigenvalues $x_j(\omega)$ of $P(t, i\omega)$, then the Cauchy problem (1.1) is well-posed. Indeed we can construct matrices $H(t,\omega)$ in the following way: if $t_0 \in (0,T)$ is any fixed value of t, then the Cauchy problem for the system

$$\partial u/\partial t = \hat{P}(t_0, \partial/\partial x)u = \delta\left(-\sum_{r=1}^{s} \partial^2/\partial x_r{}^2\right)^{m/2} u + P(t_0, \partial/\partial x)u \qquad (6.2)$$

is well-posed. There is in fact for $|\omega| \neq 0$ a unitary transformation $U(\omega)$, such that the ordering condition (4.5) is satisfied and

$$U(\omega)\hat{P}(t_0, i\omega)U^*(\omega) = U(\omega)P(t_0, i\omega)U^*(\omega) + \delta|\omega|^m I$$

$$= \begin{pmatrix} \varkappa_1 + \delta|\omega|^m & b_{12} \ldots\ldots\ldots b_{1n} \\ 0 & \varkappa_2 + \delta|\omega|^m \; b_{23} \ldots b_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & \ldots\ldots 0 \;\; \varkappa_n + \delta|\omega|^m \end{pmatrix} \qquad (6.3) \quad \underline{/117}$$

According to (6.1) the inequality

$$\mathrm{Re}(\varkappa_j + \delta|\omega|^m) \leqq -\tfrac{1}{2}\delta|\omega|^m .$$

holds for sufficiently large $|\omega|$. Since the $b_{ij}/(|\omega|^m + 1)$ are bounded it follows that the inequalities (4.6) are also satisfied for a suitable value α. Therefore, by Theorem 3 the Cauchy problem for (6.2) is well-posed. and there are constants C_4 and α and matrices $H(t_0, \omega)$ which satisfy the inequalities (4.7) so that

$$H(t_0, \omega)\hat{P}(t_0, i\omega) + \hat{P}^*(t_0, i\omega)H(t_0, \omega)$$
$$= H(t_0, \omega)P(t_0, i\omega) + P^*(t_0, i\omega)H(t_0, \omega) + 2\delta|\omega|^m H(t_0, \omega) \qquad (6.4)$$
$$\leqq 2\alpha H(t_0, \omega) .$$

If we now consider $P(t, i\omega)$ in a suitable neighborhood of t_0 then it follows for $|\omega| \geqq 1$ that:

$$H(t_0, \omega)P(t, i\omega) + P^*(t, i\omega)H(t_0, \omega)$$
$$\leqq H(t_0, \omega)P(t_0, i\omega) + P^*(t_0, i\omega)H(t_0, \omega) +$$
$$+ 2|\omega|^m |H(t_0, \omega)(P(t_0, i\omega) - P(t, i\omega))/|\omega|^m| I$$
$$= H(t_0, \omega)P(t_0, i\omega) + P^*(t_0, i\omega)H(t_0, \omega) + \delta|\omega|^m H(t_0, \omega) +$$
$$+ 2|\omega|^m \left(|H(t_0, \omega)(P(t_0, i\omega) - P(t, i\omega))/|\omega|^m| - \tfrac{1}{2}\delta H(t_0, \omega)\right)$$

12

Since the coefficients of $P(t,i\omega)$ are continuous with respect to t, it follows that there is a neighborhood of t_0 independent of ω such that

$$\left| H(t_0,\omega)(P(t_0,i\omega)-P(t,i\omega))/|\omega|^m \right| - \tfrac{1}{3}\delta H(t_0,\omega) \leq 0.$$

For this neighborhood there is therefore by (6.4) for all $|\omega|$ a constant α', such that

$$H(t_0,\omega)P(t,i\omega)+P^*(t,i\omega)H(t_0,\omega) \leq 2\alpha'H(t_0,\omega)-\delta|\omega|^mH(t_0,\omega).\qquad (6.5)$$

This shows that the Cauchy problem is locally properly posed. It then follows in the usual fashion that the Cauchy problem is also globally properly posed since each finite t-interval (0, T) can be covered by a finite number of neighborhoods of the above kind by means of the Heine-Borel theorem.

Furthermore note that one can also generalize the estimate (6.5) to the case in which the coefficients of the differential operator P depend on x also. From this fact one can then derive "a priori estimates" which lead to existence theorems. (Regarding this matter cf. Garding [1] and Mizohata [6]).

2) Homogeneous systems of the form $\partial u/\partial t = P_m(\partial/\partial x)u$ with constant coefficients for which the eigenvalues of $P_m(i\omega)$ are pure imaginaries. We prove the

THEOREM 5. Given a system

$$\partial u/\partial t = P_m(\partial/\partial x)u, \qquad P_m = \sum_{|\nu|=m} A_\nu \partial^m/\partial x_1^{\nu_1}\dots\partial x_s^{\nu_s} \qquad (6.6)$$

with constant coefficients. Then the following statements hold:

(i) If $m \equiv 1 \pmod 2$ and if the Cauchy problem is well-posed, then the eigenvalues $\kappa_j(\omega)$ of $P_m(i\omega)$ are necessarily pure imaginaries.

13

(ii) <u>If the $\varkappa_j(\omega)$ are pure imaginaries then the Cauchy problem</u> <u>is well-posed</u> <u>if and only if there is a constant</u> C_{31} <u>and to each</u> ω <u>there corresponds a matrix $S(\omega)$ with</u>

$$\max(|S|, |S^{-1}|) \leqq C_{31}$$ (6.7)

<u>so that</u>

$$S(\omega)P_m(i\omega)S^{-1}(\omega) = \begin{pmatrix} i\varkappa_1 & 0 & \ldots\ldots & 0 \\ 0 & i\varkappa_2 & 0 & \ldots & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & \ldots\ldots & 0 & i\varkappa_n \end{pmatrix}.$$ (6.8)

PROOF. For systems of the form (6.6) we have

$$P_m(i\omega) = |\omega|^m P(i\omega/|\omega|); \quad \text{i.e. } \varkappa_j(\omega) = |\omega|^m \varkappa_j(\omega/|\omega|).$$ (6.9)

Moreover if $m \equiv 1 \pmod 2$ then we also have

$$P_m(-i\omega) = -P_m(i\omega); \quad \text{i.e. } \varkappa_j(-\omega) = -\varkappa_j(\omega).$$ (6.10)

Thus for $m \equiv 1 \pmod 2$ the inequality (4.5) is satisfied precisely when $\operatorname{Re}\varkappa_j(\omega) = 0$. This proves the first statement of the theorem. The second is proved as follows. By Theorem 3 the Cauchy problem is obviously well-posed for the differential equation (6.6) if (6.7) and (6.8) hold. Conversely if the Cauchy problem is well-posed then by Theorem 3 there are matrices $\hat{S}(\omega)$ which satisfy (4.4) so that for all ω

$$\hat{S}(\omega)P_m(i\omega)\hat{S}^{-1}(\omega) = |\omega|^m \hat{S}(\omega)P_m(i\omega/|\omega|)\hat{S}^{-1}(\omega)$$

$$= |\omega|^m \begin{pmatrix} \varkappa_1' & b'_{12} & \ldots\ldots & b'_{1n} \\ 0 & \varkappa_2' & b'_{23} & \ldots & b'_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & \ldots\ldots & 0 & \varkappa_n' \end{pmatrix},$$ (6.11)

$$|\omega|^m |b'_{ij}| \leqq C_{32} |\alpha|.$$

If we now hold $\omega/\|\omega\|$ fixed and consider the sequence $\|\omega\|=1,2,\ldots,$ then we obtain from (6.11) $\lim b'_n=0$. Since the $\hat{S}(\omega)$ are uniformly bounded, we can choose a subsequence $|\omega|,$ such that $\lim \hat{S}(\omega)=S(\omega)$

exists. For the matrices $S(\omega)$ thus constructed, (6.7) and (6.8) /119

hold. This proves the second statement of the theorem.

As matrices H we can then choose

$$H = H(\omega) = S^*(\omega)S(\omega) \tag{6.12}$$

Then in fact

$$\begin{aligned} H(\omega)P_m(i\omega) &+ P_m{}^*(i\omega)H(\omega) \\ &= S^*(\omega)\big(S(\omega)P_m S^{-1}(\omega)+S^{*-1}(\omega)P_m{}^*(i\omega)S^*(\omega)\big)S(\omega) = 0. \end{aligned} \tag{6.13}$$

holds.

Moreover, the matrices H necessarily have the form (6.12). Indeed we have (cf. also O. Tausky [9])

LEMMA 1 Corresponding to a square matrix A of order n there exists a positive definite hermitian matrix H such that HA+A*H = 0 if and only if the eigenvalues of A are pure imaginaries and A has a complete system of eigenvectors. For all such matrices H one then has H = S*S where S is any matrix for which SAS^{-1} has diagonal form, i.e., S = T^{-1} where the column vectors of T consist of n linearly independent eigenvectors of A.

PROOF. Let H be any positive definite matrix such that

$$HA+A^*H = 0. \tag{6.14}$$

Since there is clearly a non-singular matrix T_1 such that $H=T_1^{*-1}T_1^{-1},$ it follows from (6.14) that

$$T_1^{-1}AT_1 + T_1^*A T_1^{*-1} = 0. \tag{6.15}$$

15

Thus $T_1^{-1}AT_1$ is antisymmetric. Hence $T_1^{-1}AT_1$ has purely imaginary eigenvalues and n linearly independent eigenvectors. The same is therefore also true of A. From (6.15) it then follows furthermore that there exists a unitary matrix U such that

$$UT_1^{-1}AT_1U^* = T^{-1}AT, \quad T = T_1U^*,$$

has diagonal form. Hence T is the desired matrix which reduces A to diagonal form, and one obtains

$$H = T_1^{*-1}T_1^{-1} = T^{*-1}T^{-1}.$$

Conversely if there is a non-singular matrix T such that $T^{-1}AT$ has diagonal form, then in accordance with (6.13) we obtain the equation (6.14) where $H=T^{*-1}T^{-1}$ in case the eigenvalues of A are pure imaginaries. However, this proves the theorem.

If in accordance with Theorem 3 we now introduce the operator H defined by (4.10) we obtain

THEOREM 5. If the eigenvalues of $P_m(i\omega)$ are pure imaginaries, then the Cauchy problem for the differential equation (6.6) is well-posed if and only if there exists a norm (u, Hu) such that for all $u(x) \in \mathfrak{M}$

$$(P_m(\partial/\partial x)u, Hu) + (u, HP_m(\partial/\partial x)u) = 0,$$

or otherwise stated, if and only if the operator $P(\partial/\partial x)$ with domain of definition $\mathfrak{M}$ is symmetric with respect to the norm (u, Hu).

7. REPRESENTATION OF THE OPERATOR H AS THE QUOTIENT OF DIFFERENTIAL OPERATORS

We shall now state conditions such that the operator H of the last theorem can be represented as a quotient of differential operators. This representation is important if one wishes to consider differential equations whose coefficients also depend on x. In accordance with Petrovskii [8, p. 56] we introduce the following terminology.

DEFINITION 2. Given a matrix $A(\beta) = (a_{ij}(\beta)), 0 \leq i, j \leq n,$ whose elements on an open set $\mathfrak{M}$ are continuous functions of a real parameter $\beta = (\beta_1, \ldots, \beta_r)$. We say that $A(\beta)$ has simple structure on $\mathfrak{M}$ if for all $\beta \in \mathfrak{M}$, $A(\beta)$ has the following properties:

1) A has n linearly independent eigenvectors.

2) The multiplicity of the roots of the characteristic equation $f(\varkappa) = \det|A - I\varkappa| = 0$ is independent of β, i.e. $f(\varkappa)$ can be written in the form

$$f(\varkappa) = \prod_\nu (g_\nu(\varkappa))^\nu = \prod_\nu (\varkappa^{r_\nu} + q_{\nu 1}\varkappa^{r_\nu - 1} + \ldots + q_{\nu r_\nu})^\nu, \quad \sum_\nu \nu r_\nu = n, \qquad (7.1)$$

where the roots $\varkappa_{\nu\mu}$ of $g_\nu(\varkappa)$ are precisely all the roots of $f(\varkappa) = 0$ of multiplicity ν.

If in addition one assumes that the a_{ij} are r-times continuously differentiable with respect to β then the same also follows for the $q_{\nu\mu}$. According to Petrovskii [8, p. 57] the $\varkappa_{\nu\mu}$ are indeed r-times continuously differentiable since they are simple roots of $d^\nu f(\varkappa)/d\varkappa^\nu = 0$.

We now prove the decisive

LEMMA 2. Let $A(\beta) = (a_{ij}(\beta))$, $0 \le i, j \le n$, be a matrix whose elements are continuous functions of a real parameter β on an open set $\mathfrak{R}$. Let A have simple structure on $\mathfrak{R}$ and let

$$\text{Re} \varkappa = 0 \,.\qquad\qquad (7.2)\qquad \underline{/121}$$

hold for the eigenvalues $\varkappa$ of A for all $\beta \in \mathfrak{R}$. Then for every $\beta^{(0)} \in \mathfrak{R}$ there is an $\varepsilon > 0$, and there is defined on $\mathfrak{R}$ a positive semi-definite hermitian matrix H(β) such that for all β with $|\beta - \beta^{(0)}| < \varepsilon$ the matrix H(β) is positive definite and

$$HA + A^*H = 0$$

Here the elements of H are integral rational functions in the $a_{ij}, \bar{a}_{ij}$ and $q_{\nu\mu}$.

PROOF. We consider any point $\beta^{(0)} \in \mathfrak{R}$. If the multiplicity of an eigenvalue $\varkappa$ equals n, then $A = \varkappa I$ and the assertion of the lemma is obviously correct. Hence we can assume that the multiplicity of all eigenvalues is less than n. We now consider all eigenvalues $\varkappa = \varkappa_{\nu\mu}$ with multiplicity $\nu < n$ and we shall determine the corresponding eigenvectors. The components x_i of these eigenvectors are determined by the linear equations

$$L_k = a_{k1}x_1 + a_{k2}x_2 + \ldots + (a_{kk} - \varkappa_{\nu\mu})x_k + \ldots + a_{kn}x_n = 0,$$
$$k = 1, \ldots, n, \qquad (7.3)$$

By hypothesis this system of equations has rank n $- \nu$, i.e., corresponding to each eigenvalue $\varkappa_{\nu\mu}$ there are exactly ν linearly independent eigenvectors. Hence we can choose constants $\lambda_{l1}, \ldots, \lambda_{ln}; l = 1, 2, \ldots, n,$ such that for $\beta = \beta^{(0)}$ the eigenvectors assigned to any one of the eigenvalues $\varkappa_{\nu\mu}$ are determined by the equations

$$G_l = \sum_{k=1}^{n} \lambda_{lk}L_k = b_{l1}x_1 + \ldots + b_{ln}x_n = 0, \qquad l = 1, 2, \ldots, n - \nu, \qquad (7.4)$$

18

The subdeterminants det $|b_{kr}|$ of order n-ν are in fact integral rational functions in the λ_{lk}, which certainly do not all vanish identically. Hence they vanish simultaneously only for special values of the λ_{lk}. Then we obtain the eigenvectors in the following way. Corresponding to the subdeterminant $h^{(1)}=\det|b_{kr}|$, $\lfloor 1 \leq k,\ r \leq n-\nu \rfloor$ of order n-ν we can find ν eigenvectors.

$$h_1^{(1)} = (h_{11},\ldots,h_{1n-},h^{(1)},0,\ldots,0)',$$
$$h_2^{(1)} = (h_{21},\ldots,h_{2n-},0,h^{(1)},0,\ldots,0)',$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

with the help of Cramer's rule. Here the h_{ij} are subdeterminants det $|b_{kr}|$ of order n-ν which have an appropriate sign. These eigenvectors are linearly independent precisely when $h^{(1)} \neq 0$. Correspondingly we can construct for each subdeterminant $h^{(i)} =$ det $|b_{kr}|$ of order n-ν, ν eigenvectors which are linearly independent exactly when $h^{(i)} \neq 0$. Since not all $h^{(i)}$ vanish /122 for every eigenvalue $|\varkappa_{\nu\mu}|$ we can, just as above, fix constants μ_{ij} such that for $\beta=\beta^{(0)}$ the ν eigenvectors

$$f_j(\varkappa_{\nu\mu}) = \sum_i \mu_{ij}h_j^{(i)}, \qquad j = 1,2,\ldots,\nu,$$

are linearly independent for all $\varkappa_{\nu\mu}$. Hence we can represent in the form

$$f_j(\varkappa_{\nu\mu}) = (f_{j1}(\varkappa_{\nu\mu}),\ldots,f_{jn}(\varkappa_{\nu\mu}))', \qquad j = 1,2,\ldots,\nu, \tag{7.5}$$

the linearly independent eigenvectors which for $\beta=\beta^0$ correspond to any of the $\varkappa_{\nu\mu}$ and ν. Here the components $f_{jk}(\varkappa_{\nu\mu})$ are integral rational functions in the a_{ij} and the $\varkappa_{\nu\mu}$. Furthermore this representation of the eigenvectors holds for all $\beta \in \mathfrak{R}$ for which the vectors (7.5) are linearly independent.

With the help of these eigenvectors we now form the matrix

$$T_\nu = (f_1(\varkappa_{\nu 1}) \ldots f_\nu(\varkappa_{\nu 1}) f_1(\varkappa_{\nu 2}) \ldots f_\nu(\varkappa_{\nu 2}) \ldots). \tag{7.6}$$

19

Here the $\varkappa_{\nu\mu}$ range over all eigenvalues of multiplicity ν, i.e., all roots of the factor $g_\nu(\varkappa)$ of (7.1). For

$$T_\nu T_\nu^* = \left(\sum_\mu \sum_l f_{lr}(\varkappa_{\nu\mu}) \overline{f_{ls}(\varkappa_{\nu\mu})} \right) = (c_{rs}), \qquad 0 \le r, \ s \le n_\nu,$$

it then follows that the c_{rs} are integral rational functions in the a_{ij}, $\overline{a}_{ij}$ and the $\varkappa_{\nu\mu}$ which are furthermore symmetric functions with respect to the $\varkappa_{\nu\mu}$. (Note that $\varkappa_{\nu\mu} = -\overline{\varkappa}_{\nu\mu}$ by (7.2)). Hence it follows from the fundamental theorem for symmetric functions that the c_{rs} can be written as integral rational functions in the a_{ij}, $\overline{a}_{ij}$ and the coefficients $q_{\nu\mu}$ of the factor $g_\nu(\varkappa)$ of (7.2). With the help of all eigenvectors of A we now form the matrix

$$T = (T_1 \ldots T_\sigma)$$ (7.7)

which is non-singular for $\beta = \beta^{(0)}$. Here σ is the highest multiplicity of the eigenvalues which occurs. Then for $\beta = \beta^{(0)}$ as well as for all $\beta \in \mathfrak{R}$, for which T is non-singular

$$T^{-1}AT = \begin{pmatrix} \varkappa_1 & 0 \ldots\ldots 0 \\ 0 & \varkappa_2 \ 0 \ldots 0 \\ \cdots\cdots\cdots\cdots \\ 0 \ldots\ldots 0 \ \varkappa_n \end{pmatrix},$$ (7.8)

and the elements of

$$TT^* = \sum_{i=1}^{\sigma} T_i T_i^*$$ (7.9)

are integral rational functions in the a_{ij}, $\overline{a}_{ij}$ and $q_{\nu\mu}$. The same is then true also for the coefficients of the hermitian matrix

$$H = \det|TT^*|(TT^*)^{-1} = (\det|T|)^2 T^{*-1} T^{-1}.$$ (7.10)

20

which is positive semi-definite for all $\beta \in \mathfrak{R}$, and positive definite at least in one neighborhood of $\beta = \beta^{(0)}$. It should be noted that for $\beta = \beta^{(0)}$, H is non-singular and that the elements of H are continuous functions of β. For all $\beta \in \mathfrak{R}$ for which H is non-singular we then obtain

$$HA + A^*H = (\det|T|)^2 T^{*-1}(T^{-1}AT + T^*A^*T^{*-1})T^1 = 0 \quad (7.11)$$

with the aid of (7.2) and (7.8). However, this proves the lemma. We next prove an important special case.

LEMMA 2'. <u>In addition to the assumptions of Lemma 2 we assume that one can split up β into $\beta_I = (\beta_1, \ldots, \beta_r)$ and $\beta_{II} = (\beta_{I+1}, \ldots, \beta_r)$ so that one can write $\mathfrak{R} = \mathfrak{R}(\beta)$ as a direct product product $\mathfrak{R}_I(\beta_I) \times \mathfrak{R}_{II}(\beta_{II})$. Moreover let the coefficients of the matrix A and the $q_{\nu\mu}$ be analytic functions of β_{II} for each fixed value β_I. Then there is for each $\beta_I^{(0)} \in \mathfrak{R}_I(\beta_I)$ an $\varepsilon > 0$ such that for all β_I with $|\beta_I - \beta_I^{(0)}| < \varepsilon$ and all $\beta_{II} \in \mathfrak{R} \subset \mathfrak{R}_{II}(\beta_{II})$ there exists a positive definite matrix H of the above kind. Here $\mathfrak{R}$ is an arbitrary compact subset of $\mathfrak{R}_{II}(\beta_{II})$.</u>

PROOF. Let $\beta_I^{(0)} \in \mathfrak{R}_I(\beta_I)$ be fixed in advance. By Lemma 2 there is for each point $\beta^{(i)} = (\beta_I^{(0)}, \beta_{II}^{(i)}) \in \mathfrak{R}(\beta)$ an $\varepsilon_i > 0$ so that for all β with $|\beta_I - \beta_I^{(0)}| < \varepsilon_i$ and $|\beta_{II} - \beta_{II}^{(i)}| < \varepsilon_i$ such a matrix H_i exists.(7.11) holds for the matrix H_i not only when $|\beta_I - \beta_I^{(0)}| < \varepsilon_i, |\beta_{II} - \beta_{II}^{(i)}| < \varepsilon_i$, but also for all $\beta \in \mathfrak{R}$ with $|\beta_I - \beta_I^{(0)}| < \varepsilon_i$. This is the case since for each fixed value β_I the elements of $H_i A + A^*H_i$ are analytic functions of β_{II} which vanish in a neighborhood of $\beta_{II}^{(i)}$. Hence they vanish identically. Then if $\mathfrak{R} \subset \mathfrak{R}_{II}(\beta_{II})$ is a compact set one can cover it with a finite number of such neighborhoods $|\beta_{II} - \beta_{II}^{(i)}| < \varepsilon_i$ and we obtain the desired matrix in the form $H = \Sigma H_i$.

Now we can prove the following theorem.

Theorem 6. <u>Given a system of differential equations</u>

$$\partial u/\partial t = P_m(\partial/\partial x)u \qquad (7.12)$$

<u>with constant coefficients. Suppose that for the eigenvalues</u> <u>of $P_m(i\omega)$, $\mathrm{Re}\,\varkappa(\omega)=0$. If $P_m(i\omega)$ has simple structure on the set</u> <u>$|\omega|=1$ then there exists a differential operator $Q(\partial/\partial x)$ of an order</u> <u>2N such that</u>

$$\delta_1\left(\sum_{\nu=1}^{s}\|\partial^N u/\partial x_\nu{}^N\|^2+\|u\|^2\right) \le (u,Q(\partial/\partial x)u) \le \delta_2\left(\sum_{\nu=1}^{s}\|\partial^N u,\partial x_\nu{}^N\|^2+\|u\|^2\right) \qquad (7.13)$$

<u>and</u>

$$(P(\partial/\partial x)u,Q(\partial/\partial x)u)+(u,Q(\partial/\partial x)P(\partial/\partial x)u) \le \delta_3(u,Q(\partial/\partial x)u) \qquad (7.14)$$

<u>for all $u\in\mathfrak{M}$. Here the δ_i are constants with $0<\delta_1\le\delta_2$</u> <u>which are independent of u.</u>

PROOF. $P(i\omega)$ has simple structure on every set $\alpha_1\le|\omega|\le\alpha_2$, $0<\alpha_1<1<\alpha_2$. This follows directly from the equation

$$P_m(i\omega) = |\omega|^m P_m(i\omega/|\omega|) . \qquad (7.15)$$

Therefore, the $g_{\nu\mu}$ are integral rational functions of ω. For the factors $g_\nu(\varkappa)$ are the common divisor of $f(\varkappa)$ and $d'f(\varkappa)/d\varkappa$. Hence the representation (7.1) holds in the ring of polynomials $P(\omega_1,\ldots,\omega_s,\varkappa)$ over the field of complex numbers. If we therefore apply Lemma 2' to such a set with $\beta_{II}=\omega$ then there exists for $|\omega|=1$ a positive definite hermetian matrix $H_1(\omega)$ whose elements are integral rational functions in the ω such that

$$P_m{}^*(i\omega)H_1(\omega)+H_1(\omega)P_m(i\omega) = 0$$

Since $P_m(-i\omega)=(-1)^m P_m(i\omega)$ we also have

$$P_m{}^*(i\omega)H_1(-\omega)+H_1(-\omega)P_m(i\omega) = 0 .$$

Thus if we form the matrix $H_2(\omega)=H_1(-\omega)+H_1(\omega),$ then there exist

constants σ_1, σ_2 with $0 < \sigma_1 \leqq \sigma_2$, such that for $|\omega| = 1$

$$\sigma_1 I \leqq H_2(\omega) \leqq \sigma_2 I , \tag{7.16}$$

$$P_m^*(i\omega) H_2(\omega) + H_2(\omega) P_m(i\omega) = 0 . \tag{7.17}$$

Moreover the elements of $H_2(\omega)$ are even integral rational functions in the components of ω. We denote by ord H_2 the highest occurring order of these functions. We now define H_2 for arbitrary ω by means of

$$H_{\hat{2}}(\omega) = |\omega|^{2N} H_2(\omega/|\omega|), \qquad N = \max([\tfrac{1}{2}(m+1)], \tfrac{1}{2} \text{ord} H_2) . \tag{7.18}$$

([x] denotes the largest integer $\leqq x$). Then it follows from (7.15) that (7.17) holds for all ω, and from (7.16) that

$$\sigma_1 |\omega|^{2N} I \leqq H_2(\omega) \leqq \sigma_2 |\omega|^{2N} I . \tag{7.19}$$

Furthermore, the elements of $H_2(\omega)$ are polynomials in the com- /125 ponents of ω. If we now let

$$H(\omega) = H_2(\omega) + \sigma_2 I , \tag{7.20}$$

then it follows from (7.19) and (7.20) and $2N > m$ that

$$\sigma_1 (|\omega|^{2N} + 1) I \leqq H(\omega) \leqq \sigma_2 (|\omega|^{2N} + 1) I , \tag{7.21}$$

$$P_m^*(i\omega) H(\omega) + H(\omega) P_m(i\omega) = \sigma_2 (P_m(i\omega) + P_m^*(i\omega)) \tag{7.22}$$
$$\leqq \text{const.} \ (|\omega|^m + 1) \leqq \text{const.} \ H(\omega) .$$

Next we understand by $Q(\partial/\partial x)$ the differential operator of order $2N$ for which $Q(i\omega) = H(\omega)$. If one observes that for all $u \in \mathfrak{M}$

$$\sum_{\nu=1}^{s} \|\partial^N u/\partial x_\nu^N\|^2 + \|u\|^2 = \int_{R_s} \left(\sum_{\nu=1}^{s} |\omega_\nu|^{2N} + 1 \right) |\psi(\omega)|^2 \, d\omega ,$$

($\psi(\omega)$ is the Fourier transform of u), then it follows from (7.21) that for $Q(\partial/\partial x)$ an inequality (7.13) holds. Furthermore (7.14) follows just like (4.11). From this fact the assertion follows.

If we now set

$$R(\partial/\partial x) = \left(-\sum_{\nu=1}^{s} \partial^2/\partial x_\nu{}^2 \right)^N,$$

then for all $u \in \mathfrak{M}$,

$$R(\partial/\partial x)u = (2\pi)^{-s/2} \int_{R^s} |\omega|^{2N} e^{i\omega x} \psi(\omega) \, d\omega .$$

If we note that by (7.20)

$$(Q(\partial/\partial x)-I)u = (2\pi)^{-s/2} \int_{R^s} H_2(\omega) e^{i\omega x} \psi(\omega) \, d\omega ,$$

then by (7.16) and (7.18) we can define a bounded positive definite symmetric operator

$$H = R^{-1}(\partial/\partial x)(Q(\partial/\partial x)-I) \quad \text{for all} \quad u \in L_2$$

by

$$Hu = \int_{R^s} H_2(\omega) |\omega|^{-2N} e^{i\omega x} \psi(\omega) \, d\omega$$

$$= \int_{R^s} H_2(\omega/|\omega|) e^{i\omega x} \psi(\omega) \, d\omega$$

for which by (7.17) for all $u \in \mathfrak{M}$

$$(P_m(\partial/\partial x)u, Hu) + (u, HP_m(\partial/\partial x)u) = 0 .$$

Therefore in this case one can represent the operator H of Theorem 3 as the quotient of differential operators.

8. GENERALIZATION TO SYSTEMS OF FIRST ORDER DIFFERENTIAL EQUATIONS WHOSE COEFFICIENTS DEPEND ON x AND t.

In this section we wish to prove the following theorem.

THEOREM 7. <u>Given a system of first order differential equations</u>

$$\partial u/\partial t = \sum_{\nu=1}^{e} A_\nu(x,t)\partial u/\partial x_\nu = P_1(x,t,\partial/\partial x)u ,$$

<u>whose coefficients are infinitely (sufficiently) differentiable in all variables. We assume that for all real ω and all x and t the matrices $P(x,t,i\omega)=\sum_\nu A_\nu(x,t)i\omega_\nu$ have purely imaginary eigenvalues and have simple structure. Then for each point (x_0, t_0) there is a neighborhood of U and a formally self-adjoint differential operator $Q(x,t,\partial/\partial x)$ of even order 2N>0 whose coefficients are infinitely (sufficiently) differentiable such that for all $u(x,t)\in \bar{C}_0^\infty(U)$ the inequalities (7.13) - (7.14) hold if one replaces $Q(\partial/\partial x)$ by $Q(x,t,\partial/\partial x)$. [$\bar{C}_0^\infty(U)$ is the class of all infinitely differentiable functions which vanish outside of a compact set $\subset U$].</u>

PROOF. By Lemma 2' and according to the proof of Theorem 6 there is for each point (x_0, t_0) a neighborhood U in which there exists an hermitian matrix $H(x,t,i\omega)$ corresponding to $P(x,t,i\omega)$ whose elements are integral rational functions of the ω_ν with coefficients which are infinitely differentiable with respect to x and t (Cf. the remark in connection with definition 2) so that for all $(x,t)\in U$ the inequalities corresponding to (7.21) and (7.22) hold. Hence we can construct a formally self-adjoint differential operator $Q(x,t,\partial/\partial x)$ for which

$$Q(x,t,i\omega) = H(x,t,\omega) .$$

(8.1)

Then by (7.21) for each point $(x_1, t_1) \in U$ the inequality (7.13) holds for the differential operator $Q(x_1, t_1, \partial/\partial x)$ when one replaces $Q(\partial/\partial x)$ by $Q(x_1, t_1, \partial/\partial x)$. According to Garding [1], (7.13) holds for appropriate $\delta_1 > 0$, $\delta_2 > 0$ for the differential operator $Q(x, t, \partial/\partial x)$ also, i.e.

$$\delta_1 \left(\sum_{\nu=1}^{s} \|\partial^N u/\partial x^N\|^2 + \|u\|^2 \right) \leqq (u, Q(x, t, \partial/\partial x)u)$$
$$\leqq \delta_2 \left(\sum_{\nu=1}^{s} \|\partial^N u/\partial x^N\|^2 + \|u\|^2 \right), \tag{8.2}$$

if $u \in C_0^\infty(U)$ and U is sufficiently small.

If we now consider
$$L(x, t, \partial/\partial x)$$
$$= P_1{}^*(x, t, \partial/\partial x)Q(x, t, \partial/\partial x) + Q(x, t, \partial/\partial x)P_1(x, t, \partial/\partial x)u, \tag{8.3}$$

where $P_1{}^*(x, t, \partial/\partial x)$ is the operator formally adjoint to $P_1(x, t, \partial/\partial x)$ then it follows from (8.1) and the equation corresponding to (7.22) that

$$L(x, t, i\omega) = P_1{}^*(x, t, i\omega)Q(x, t, i\omega) + Q(x, t, i\omega)P_1(x, t, i\omega)$$
$$= \sigma_2(P_1{}^*(x, t, i\omega) + P_1(x, t, i\omega)) .$$

Hence the differential operator $L(x, t, \partial/\partial x)$ is at most of order 2N. If $\underline{a}$ is a constant vector then we have

$$L(x, t, \partial/\partial x)ae^{i\omega x} = (L(x, t, i\omega) + G(x, t, i\omega))ae^{i\omega x}, \tag{8.4}$$

where the elements of $G(x, t, i\omega)$ are polynomials in the ω_ν whose order is less than or equal to 2N. Hence it follows for all $u \in C_0^\infty(U)$ by means of integration by parts that

$$(u, L(x, t, \partial/\partial x)u) = (L_1(x, t, \partial/\partial x)u, L_2(x, t, \partial/\partial x)u),$$

where the L_i are differential operators of order less than or equal to N. Hence by (8.2) one can estimate the expression $\left|(u,L(x,t,\partial/\partial x)u)\right|$ by means of $(u,Q(x,t,\partial/\partial x)u)\left|\right.$ and obtain

$$\left|(P_1(x,t,\partial/\partial x)u, Q(x,t,\partial/\partial x)u) + (u,Q(x,t,\partial/\partial x)P_1(x,t,\partial/\partial x)u)\right|$$
$$= (u,L(x,t,\partial/\partial x)u) \leq const.\cdot(u,Q(x,t,\partial/\partial x)u). \qquad (8.5)$$

But this proves the theorem.

In accordance with (4.12) it follows from this theorem that there is an "a priori" estimate of the local solutions and thus by means of standard methods (cf. e.g. Leray [5]) that there exist global solutions, e.g. for all sufficiently differentiable initial values.

Thus with the help of the last theorem one can generalize the results of I. Petrovskii [7]. Indeed Petrovskii requires that the eigenvalues of $P(x,t,\partial/\partial x)$ must be distinct for all x,t,ω. Then these matrices are certainly of simple structure.

REFERENCES

1. Garding, L. Dirichlet's Problem for Linear Elliptic Partial Differential Equations. Math. Scand. Vol. 1, 1953, pp. 55-72.

2. Garding, L. Linear Hyperbolic Partial Differential Equations with Constant Coefficients. Acta Math. Vol. 85, 1951, pp. 1-62.

3. Hadamard, J. Lectures on Cauchy's Problem in Linear Partial Differential Equations, Yale, 1921.

4.   Kreiss, H. O.   On Matrices which Generate Bounded Semi-Groups.
     Math. Scand.   Vol. 7, 1959, pp. 71-80.

5.   Leray, J.   Hyperbolic Differential Equations.   Princeton.   1953.

6.   Mizohata, S.   The Cauchy Problem for Parabolic Equations.
     J. Math. Soc. Japan, Vol. 8, 1956, pp. 269-299.

7.   Petrovskii, I. G.   On Cauchy's Problem for Systems of
     Partial Differential Equations.   Math. Sbornik N. S.
     Vol. 44, 1937, pp. 814-868.

8.   Petrovskii, I. G.   On the Cauchy Problem for a System of
     Linear Partial Differential Equations in the Realm of
     Non-Analytic Functions. Bull. Univ. Etat Moscow,
     Ser. Int. Sect. A Fasc. Vol. 7, 1938, pp. 1-74.

9.   Tausky, O.   Solution of a Proposed Problem.   Amer. Math.
     Monthly, Vol. 67, 1960, p. 192.